

Integrable Spin Chain and Operator Mixing in $\mathcal{N} = 1, 2$ Supersymmetric Theories

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Abstract

We study operator mixing, due to planar one-loop corrections, for composite operators in $D = 4$ supersymmetric theories. We present some $\mathcal{N} = 1, 2$ Yang-Mills and Wess-Zumino models, in which the planar one-loop anomalous dimension matrix in the sector of holomorphic scalars is identified with the Hamiltonian of an integrable quantum spin chain with $SU(3)$ or $SU(2)$ symmetry, even if the theory is away from the conformal points. This points to a more universal origin of the integrable structure beyond superconformal symmetry. We also emphasize the role of the superpotential in the appearance of the integrable structure. The computations of operator mixing in our examples by solving Bethe Ansatz equations show some new features absent in $\mathcal{N} = 4$ SYM.

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I. INTRODUCTION

Whether the integrable structure that is abundant in two dimensional field theories or statistical models could emerge in four dimensions has been a fascinating topic on the research frontier. Indeed, a (non-topological) Yang-Mills theory in four dimensions should have very rich physics, so that it is hard to believe it could be exactly solved as a whole the same way as in two dimensions. On the other hand, one cannot either rule out the possibility that an integrable structure appears in a subsector or in a special limit. Actually, there have been quite a few evidences for an integrable structure in the self-dual sector of a Yang-Mills theory [1, 2, 3, 4, 5, 6, 7] or in the high energy limit of QCD [8, 9, 10].

Recently there have been revived interests in the integrable structure in $\mathcal{N} = 4$ superconformal Yang-Mills (SCYM) theory in four dimensions, at least in the planar limit. One development [11] was to apply the integrable structure, originally found in the QCD context [12], to the computation of anomalous dimensions of twist-two operators in $\mathcal{N} = 4$ SCYM. (This result was also obtained by means of other methods [13].) Another strong evidence came from a recent one-loop calculation of the anomalous dimensions of composite operators [14]. (The interest in composite operators in the gauge theory was mainly inspired by the proposal [15] that at least some of them could be viewed as the holographic dual of string states in a special limit of the curved background $AdS_5 \times S^5$.) It turned out [14] that the planar one-loop mixing matrix for the composite operators, that consist of a string of scalar fields in the theory, is equivalent to the Hamiltonian of an integrable spin chain with $SO(6)$ symmetry! One may naturally ask whether a similar integrable structure could appear in other Yang-Mills theories, or Wess-Zumino models, with less supersymmetries or even with conformal symmetry broken. This is an interesting issue, because one wants to know whether the appearance of the integrable structure in the $\mathcal{N} = 4$ SCYM is related to the maximal superconformal symmetry that the theory has.

In this paper, we present examples in which a similar integrable structure survives deformation of the theory with less unbroken supersymmetries. More concretely, we show that in some $\mathcal{N} = 1$ (and $\mathcal{N} = 2$) supersymmetric Yang-Mills theories in four dimensions, the planar one-loop anomalous dimension matrix for composite operators of holomorphic scalars is equivalent to the Hamiltonian of an integrable quantum spin chain, even if the theory is *away from* the conformal points. If we take a limit in which the 't Hooft gauge

couplings vanish, these models reduce to Wess-Zumino models, and the integrable structure still survives this limit. Our results indicate that the appearance of the integrable structure in Yang-Mills theory in four dimensions should have a more universal and profound origin, not restricted only to theories with (maximal) superconformal symmetry. Moreover, in our examples, the superpotential term is seen to play a crucial role in determining the integrable structure.

Some of our models are deformation of orbifolding daughters of the $\mathcal{N} = 4$ SCYM with the superpotential strength changed. In their conformal phase, they are reduced to a quiver theory with a product gauge group [16] obtained by orbifolding the $\mathcal{N} = 4$ SCYM with a discrete subgroup of $SO(6)$ R -symmetry. We therefore expect that there is a close relationship between integrable spin chains in these $\mathcal{N} = 1, 2$ SYM and the $SO(6)$ chain appearing in the $\mathcal{N} = 4$ SCYM. Indeed it turns out that all the integrable spin chains revealed by us in these $\mathcal{N} = 1, 2$ SYM is closely related to a closed subsector of an $SO(6)$ chain; at conformal points the parameters of the latter reduce to those of the $SO(6)$ chain in $\mathcal{N} = 4$ SCYM. However, the operator mixing that results from diagonalizing the spin Hamiltonian is completely different, because of an interesting interplay between the global symmetry index and the discrete index for gauge group factors, which is absent in the $\mathcal{N} = 4$ SYM.

The results for our $\mathcal{N} = 1, 2$ models, combined together, provide a description of cascade breaking of the symmetry of the integrable spin chains starting from the $\mathcal{N} = 4$ SYM. This motivates to make the conjecture that in *all* $\mathcal{N} = 1, 2$ orbifolded daughters [18, 19] of the $\mathcal{N} = 4$ SYM (with a non-abelian global symmetry), with their 't Hooft and superpotential couplings deformed away from the conformal points while keeping the global symmetry of the superpotential, there is always a non-trivial *integrable structure* in operator mixing at planar one-loop level for (anti-) holomorphic composite operators, at least for those consisting of purely scalars without derivatives. Moreover, this integrable structure survives in the resulting Wess-Zumino models when all 't Hooft couplings are sent to zero.

The contents of this paper are as follows. In section 2 we construct an $\mathcal{N} = 1$ SYM model whose conformal phase is the orbifolding limit of $\mathcal{N} = 4$ SCYM. In section 3 we compute the planar one-loop corrections to composite operators consisting of scalar fields of the model. In section 4 we obtain the mixing matrix for the renormalized composite operators, and show that the anomalous dimension matrix in the (anti-)holomorphic sector is equivalent to a Hamiltonian of an integrable $SU(3)$ spin chain. We generalize our discussion to other

$\mathcal{N} = 1$ (and $\mathcal{N} = 2$) Yang-Mills and Wess-Zumino models in section 5, and devote section 6 to a brief summary. In two appendixes we present examples for computing planar one-loop anomalous dimensions via solving the Bethe ansatz equations of the $SU(3)$ quantum spin chain in section 4 for our $\mathcal{N} = 1$ model. This computation will explicitly demonstrate how and why the operator mixing differs from the $\mathcal{N} = 4$ SYM case, though the quantum spin chain can be viewed as a closed subsector of the latter.

II. THE $\mathcal{N} = 1$ MODEL

Various $\mathcal{N} = 1$ superconformal gauge theories have been constructed [18, 19] via considering D3-branes at orbifold singularities of the form \mathbb{C}^3/Γ (with Γ a discrete subgroup of the $SO(6)$ R-symmetry of $\mathcal{N} = 4$ SYM) [16, 20]. In AdS/CFT correspondence [21], the Γ action is translated to an action $AdS_5 \times (S^5/\Gamma)$ with the AdS part unaffected. So the world-volume theory on N D3-branes remains to be a conformal field theory. With appropriately chosen Γ , supersymmetry is broken down to $\mathcal{N} = 2, 1$ or 0.

The simplest example for $\mathcal{N} = 1$ SCYM is the case with $\Gamma = \mathbb{Z}_3$ proposed in ref. [18]. The gauge group of the resulting theory has a product group $U(N)^{(1)} \times U(N)^{(2)} \times U(N)^{(3)}$. We will use a discrete index $A = 1, 2, 3$ to label these $U(N)$ groups. The matter in the theory consists of the bi-fundamental chiral superfields

$$3\{(N, \bar{N}, 1) \oplus (\bar{N}, 1, N) \oplus (1, N, \bar{N})\}, \quad (1)$$

and their anti-chiral partners (Fig. 1). Inside each pair of circular brackets, we have the representations of the gauge groups, while the overall factor 3 in Eq. (1) reflects an $SU(3)$ global symmetry inherited from the $SO(6)$ symmetry of the $\mathcal{N} = 4$ SYM, that is broken down to $SU(3)$ by orbifolding.

Therefore, each of the $U(N)$ groups is coupled to $N_f = 3N$ fundamental chiral and $N_{\bar{f}} = 3N$ anti-fundamental anti-chiral matter. One may denote the bi-fundamental chiral matter fields in Eq. (1) as $\Phi^{a,(A\bar{B})}$. Here the pair of indices, (A, \bar{B}) , labels which two of the gauge groups that the field is coupled to; and the index $a = 1, 2, 3$ is that for the representation **3** of $SU(3)$. To simplify the notation, we will write $\Phi_A^a = \Phi^{a,(B\bar{C})}$ with (A, B, C) being a cyclic permutation of $(1, 2, 3)$; and denote their Hermitian conjugate as $\tilde{\Phi}_{\bar{A}}^{\bar{a}} = (\Phi_A^a)^\dagger$ (Fig. 1), with the tilde symbol suppressed when this is no confusion.

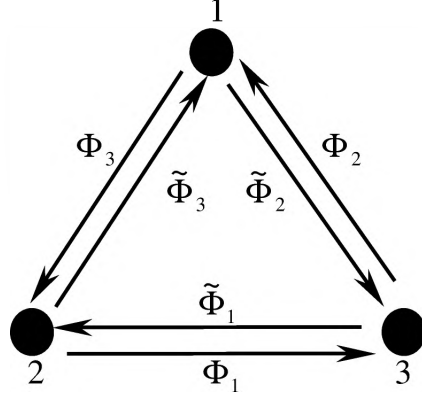


FIG. 1: A quiver diagram for D3-branes on a $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. At the nodes we have vectormultiplets in the gauge group indicated, while the arrows connecting each pair of nodes correspond to the bi-fundamental fields.

With the matter content (1), one can construct an $\mathcal{N} = 1$ gauge theory, which is not necessarily conformal invariant by allowing unequal gauge couplings and/or a more general superpotential. In the standard $\mathcal{N} = 1$ superfield formalism, the Lagrangian of our model reads

$$\begin{aligned} \mathcal{L} = & \frac{1}{4} \int d^4\theta \sum_A \text{tr}\{(\Phi_A^a)^\dagger e^{V_B} \Phi_A^a e^{-V_C}\} \\ & + \frac{1}{4} \int d^2\theta \text{tr}(W_A W_A + \text{h.c.}) + \frac{1}{2} \int d^2\theta (\mathcal{W} + \text{h.c.}). \end{aligned} \quad (2)$$

In the first term, we assumed that (A, B, C) is a cyclic permutation of $(1, 2, 3)$. Here to deform the theory obtained by orbifolding the $\mathcal{N} = 4$ SYM, we allow the gauge couplings, g_A with $A = 1, 2, 3$, of the three $U(N)$ groups to be different from each other, and take the superpotential \mathcal{W} to be a generic $SU(3)$ invariant one:

$$\mathcal{W} = \frac{h}{3} \epsilon_{abc} \text{tr} \Phi_1^a \Phi_2^b \Phi_3^c. \quad (3)$$

with h arbitrary. So the global symmetry remains to be $SU(3)$ after the deformation. If the three gauge couplings are the same, our model also has a discrete \mathbb{Z}_3 symmetry acting on the index A . The one-loop beta functions in this model can be extracted from the NSVZ beta function [22] as well as the results in ref. [23]:

$$\begin{aligned} \beta_{\lambda_A} &= \frac{d\lambda_A}{d\ln\mu} = -\frac{\lambda_A^2}{4\pi^2} \left(1 - \frac{\lambda_A}{8\pi^2}\right)^{-1} \sum_{B \neq A} \gamma_B, \\ \beta_h &= \frac{d\lambda_h}{d\ln\mu} = \lambda_h \sum_A \gamma_A, \end{aligned} \quad (4)$$

where $\lambda_A = g_A^2 N$ ($A = 1, 2, 3$) are 't Hooft couplings for the three $U(N)$ groups and $\lambda_h = |h|^2 N$; γ_A are anomalous dimensions of the three types of bi-fundamental scalar fields. In large N limit, the one-loop γ_A are

$$\gamma_A = \frac{1}{2} \frac{d \ln Z_{\phi_A}}{d \ln \mu} = \frac{1}{16\pi^2} \left(\sum_{B \neq A} \lambda_B - 2\lambda_h \right), \quad (5)$$

In this paper we will work in the region with weak 't Hooft couplings and small λ_h . Substituting Eq. (5) into Eq. (4) we have

$$\begin{aligned} \beta_{\lambda_A} &= \frac{d\lambda_A}{d \ln \mu} = -\frac{\lambda_A^2}{64\pi^4} \left(\sum_B \lambda_B + \lambda_A - 4\lambda_h \right) + O(\lambda^4), \\ \beta_h &= \frac{d\lambda_h}{d \ln \mu} = \frac{\lambda_h}{8\pi^2} \left(\sum_A \lambda_A - 3\lambda_h \right) + O(\lambda^3). \end{aligned} \quad (6)$$

There is a line of fixed points, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_h$, corresponding to the orbifolded SCYM. Though the above equations cannot be exactly solved we can consider the simplest case with $\lambda_1 = \lambda_2 = \lambda_3$, denoted as λ , to demonstrate some features for the flow. First from the flow equations it is easy to see that if initially the three gauge couplings coincide, then they will keep to do so during the renormalization group flow. Secondly we can easily see that during the flow one always has $\lambda_h = c \exp(6\pi^2/\lambda)$, with an arbitrary positive integral constant c . So for fixed value of c , both the 't Hooft and the superpotential couplings, λ and λ_h , respectively, possess a fixed point $\lambda = \lambda_h = \lambda_*(c)$, whose value is determined by the root of the transcendental equation $x = c \exp(6\pi^2/x)$. At the fixed point, the theory is an interacting conformal field theory at the quantum level. (The theory is no longer asymptotically free for finite $\lambda_*(c)$). When c varies one obtains a line of RG fixed points. When away from the conformal points, at infra-red the theory will be dominant by gauge interactions if $\lambda > \lambda_*$, $\lambda_h < \lambda_*$ and by superpotential interactions if $\lambda < \lambda_*$, $\lambda_h > \lambda_*$. In the following we will calculate the operator mixing for small values of the couplings λ_A and λ_h .

III. MATRIX OF RENORMALIZATION FACTORS

We will study one-loop renormalization of composite operators, which are a product of bi-fundamental scalars ϕ_A^a , the scalar component of the supermultiplets Φ_A^a , and their Hermitian conjugate $\phi_{\bar{A}}^{\bar{a}} = (\phi_A^a)^\dagger$, without derivatives:

$$\mathcal{O}^{I_1 \dots I_L} = \text{tr } \phi^{I_1} \dots \phi^{I_L}, \quad (7)$$

where each index I_l ($1 \leq l \leq L$) stands for a pair of indices (a_l, A_l) or \bar{a}_l, \bar{A}_l . The index I_l for a fixed l can take 18 different values. The fields at the right-hand side are all taken to be at the same space-time point. These operators form (reducible) $SU(3)$ tensors with L indices. In particular, this class of composite operators includes purely chiral (or holomorphic) operators which are a product of ϕ_A^a 's only (with no indices of type \bar{a}). In physics one is restricted to composite operators that are *gauge invariant* after taking the trace over the gauge-group matrix indices. To compute operator mixing, it is better to work with the composite operators before taking the trace. We note that gauge invariance strongly constrain possible choice of the index sequence (I_1, \dots, I_L) . (See below.) In general, the scalar operators (7) mix under renormalization, and the renormalized operators with definite dimension are linear combinations of the bare operators. If we specify a particular operator basis, \mathcal{O}^I with I being a sequence I_1, \dots, I_L , the matrix Z_J^I of renormalization factors in this basis,

$$\mathcal{O}_{\text{ren}}^I = Z_J^I \mathcal{O}^J, \quad (I = 1, \dots, 18^L), \quad (8)$$

is defined by the requirement of canceling UV divergences in the correlation functions

$$\langle Z_{\phi_{I_1}}^{1/2} \phi_{I_1}(x_1) \dots Z_{\phi_{I_L}}^{1/2} \phi_{I_L}(x_L) \mathcal{O}_{\text{ren}}^{J_1 \dots J_L}(x) \rangle, \quad (9)$$

These Z_J^I depend on the UV cutoff Λ as well as various couplings of the theory (in the large- N limit), and form a $18^L \times 18^L$ matrix. In rest of this section, we will compute the $\delta Z_J^I \equiv Z_J^I - \delta_J^I$ due to planar one-loop diagrams, using the component field formalism as in ref. [14]. There are three types of planar one-loop diagrams, as shown in Fig. 2, that contribute to the correlations (9). We will choose the Fermi-Feynman gauge for gauge boson propagators, in which the anomalous dimension of a single scalar field is already given by Eq. (5).

The one-loop self-energy diagram, Fig. 2c, leads to the wave function renormalization of scalar fields, which can be directly read off from Eq. (5). Half of the self-energy correction in correlation functions (9) are cancelled by wave function renormalization of the external legs. The counterterms that cancel the remaining divergences are given by

$$\delta Z_{\dots I_l I_{l+1} \dots}^{(c) \dots J_l J_{l+1} \dots} = -\frac{1}{2}(\gamma_{A_l} + \gamma_{A_{l+1}}) \ln \Lambda \delta_{I_l}^{J_l} \delta_{I_{l+1}}^{J_{l+1}}, \quad (10)$$

where each I_l or J_l stands for a pair of indices (i_l, A_l) or (j_l, B_l) , respectively. Note that

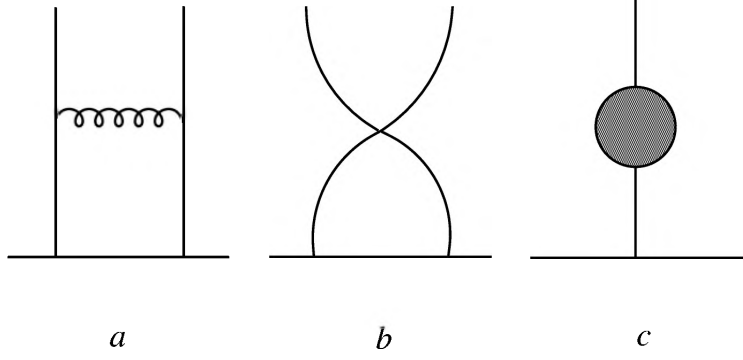


FIG. 2: One-loop planar diagrams on correlation functions (9).

the contribution of this diagram to the Z -matrix is diagonal in the indices I_l , J_l and in I_{l+1} , J_{l+1} .

As for the contributions from Fig. 2a and 2b, we have to distinguish between two different cases: i) two nearest-neighbor scalar fields both are holomorphic (or anti-holomorphic), namely $\mathcal{O} \sim \dots \phi_{A_l}^{a_l} \phi_{A_{l+1}}^{a_{l+1}} \dots$ (or $\dots \phi_{\bar{A}_l}^{\bar{a}_l} \phi_{\bar{A}_{l+1}}^{\bar{a}_{l+1}} \dots$); ii) one of them is holomorphic and the other anti-holomorphic, i.e. $\mathcal{O} \sim \dots \phi_{A_l}^{a_l} \phi_{\bar{A}_{l+1}}^{\bar{a}_{l+1}} \dots$ (or $\dots \phi_{\bar{A}_l}^{\bar{a}_l} \phi_{A_{l+1}}^{a_{l+1}} \dots$). If the nearest-neighbor pairs in a composite operator \mathcal{O} all belong to the above case i), we call it a *holomorphic* (or *anti-holomorphic*, respectively) operator. Otherwise, if the above case ii) happens to one nearest-neighbor pair, the composite operator is called *non-holomorphic*.

The correction due to gauge boson exchange, Fig. 2a, contributes to the Z -matrix associated with holomorphic operators:

$$\delta Z_{\dots I_l I_{l+1} \dots}^{(a) \dots J_l J_{l+1} \dots} = \frac{\lambda_{A_l^-}}{16\pi^2} \ln \Lambda \delta_{I_l}^{J_l} \delta_{I_{l+1}}^{J_{l+1}}, \quad (11)$$

where $A_l^\pm = (A_l \pm 1) \bmod 3$. For anti-holomorphic operators, one just replaces $A_l^- \rightarrow \bar{A}_l^+$. Meanwhile, the same one-loop diagram, Fig. 2a, yields the following Z -matrix for non-holomorphic operators:

$$\begin{aligned} \delta Z_{\dots I_l \bar{I}_{l+1} \dots}^{(a) \dots J_l \bar{J}_{l+1} \dots} &= \frac{\lambda_{A_l^-}}{16\pi^2} \ln \Lambda \delta_{I_l}^{J_l} \delta_{\bar{I}_{l+1}}^{\bar{J}_{l+1}}, \\ \delta Z_{\dots \bar{I}_l I_{l+1} \dots}^{(a) \dots \bar{J}_l J_{l+1} \dots} &= \frac{\lambda_{\bar{A}_l^+}}{16\pi^2} \ln \Lambda \delta_{\bar{I}_l}^{\bar{J}_l} \delta_{I_{l+1}}^{J_{l+1}}. \end{aligned} \quad (12)$$

Note that here an overall factor of $\delta_{A_l \bar{A}_{l+1}}$ or $\delta_{\bar{A}_l A_{l+1}}$ is implied due to the requirement of gauge invariance.

As for Fig. 2b, one has to be careful in extracting the $SU(3)$ structure of the resulting contributions to the Z -matrix, because the quartic scalar vertex in this diagram involves

both gauge and superpotential couplings. For holomorphic operators, we have

$$\delta Z^{(b) \dots J_l J_{l+1} \dots}_{\dots I_l I_{l+1} \dots} = -\frac{\ln \Lambda}{8\pi^2} \left\{ \left(\frac{\lambda_{A_l^-}}{2} - \lambda_h \right) \delta_{I_l}^{J_l} \delta_{I_{l+1}}^{J_{l+1}} + \lambda_h \delta_{I_l}^{J_{l+1}} \delta_{I_{l+1}}^{J_l} \right\}. \quad (13)$$

For non-holomorphic operators the Z -matrix exhibits some complications:

$$\begin{aligned} \delta Z^{(b) \dots J_l \bar{J}_{l+1} \dots}_{\dots I_l \bar{I}_{l+1} \dots} &= \frac{\ln \Lambda}{16\pi^2} \lambda_{A_l^-} (\delta_{I_l}^{J_l} \delta_{\bar{I}_{l+1}}^{\bar{J}_{l+1}} + \delta_{I_l, \bar{I}_{l+1}} \delta^{J_l, \bar{J}_{l+1}}), \\ \delta Z^{(b) \dots \bar{J}_l J_{l+1} \dots}_{\dots I_l \bar{I}_{l+1} \dots} &= -\frac{\ln \Lambda}{8\pi^2} \left\{ \lambda_h \delta_{I_l}^{J_{l+1}} \delta_{\bar{I}_{l+1}}^{\bar{J}_l} - \left(\lambda_h - \frac{\lambda_{A_l^-}}{2} \right) \delta_{I_l, \bar{I}_{l+1}} \delta^{\bar{J}_l, J_{l+1}} \right\}, \\ \delta Z^{(b) \dots \bar{J}_l J_{l+1} \dots}_{\dots \bar{I}_l I_{l+1} \dots} &= \frac{\ln \Lambda}{16\pi^2} \lambda_{\bar{A}_l^+} (\delta_{\bar{I}_l}^{\bar{J}_l} \delta_{I_{l+1}}^{J_{l+1}} + \delta_{\bar{I}_l, I_{l+1}} \delta^{\bar{J}_l, J_{l+1}}), \\ \delta Z^{(b) \dots \bar{J}_l J_{l+1} \dots}_{\dots \bar{I}_l I_{l+1} \dots} &= -\frac{\ln \Lambda}{8\pi^2} \left\{ \lambda_h \delta_{\bar{I}_l}^{\bar{J}_{l+1}} \delta_{I_{l+1}}^{J_l} - \left(\lambda_h - \frac{\lambda_{\bar{A}_l^+}}{2} \right) \delta_{\bar{I}_l, I_{l+1}} \delta^{\bar{J}_{l+1}, J_l} \right\}. \end{aligned} \quad (14)$$

In next section, we will consider the operator mixing resulting from above Z -factors.

IV. OPERATOR MIXING AND SPIN CHAIN

The anomalous dimension matrix (ADM) at one loop for operators (7) is determined by the standard arguments through

$$\Gamma = \frac{d\delta Z}{d \ln \Lambda}. \quad (15)$$

Operator mixing arises when one diagonalizes this matrix to obtain operators with definite dimension. The ADM acts on a 18^L -dimensional vector space $\mathcal{V}_1 \otimes \dots \otimes \mathcal{V}_L$, where each \mathcal{V}_l is a complex vector space spanned by ϕ_A^a and $\phi_{\bar{A}}^{\bar{a}}$. As proposed by Minahan and Zarembo [14] for the $\mathcal{N} = 4$ SYM, it is extremely useful to identify the ADM as the Hamiltonian of a lattice spin system, where the lattice sites are labeled by the subscript $l = 1, 2, \dots, L$. The main goal of our paper is to look for an integrable spin chain for the ADM, at least in a sector, in our model.

However, there is a technical complication here. In the $\mathcal{N} = 4$ case, the adjoint scalars are labeled only by one index $i = 1, \dots, 6$ for the $SO(6)$ symmetry. But in our model, we need a pair of indices (a, A) (or \bar{a}, \bar{A}) to label the bi-fundamental scalars, where the extra index A is necessary to label the arrows in the quiver diagram, indicating which pair of $U(N)$ gauge groups the scalar is coupled to. This is because our model is a deformation of an orbifolded model. We note, however, that in accordance with the quiver diagram, Fig.

1, it is easy to see that gauge invariance of a composite operator (7) strongly constrains its index sequence I_1, \dots, I_L . For example, if I_l is known, then gauge invariance dictates the gauge index in I_{l+1} , depending on whether its $SU(3)$ index is of type a or type \bar{a} . Thus if A_1 is known, then gauge index sequence of a gauge invariant composite operator is completely determined by its $SU(3)$ index sequence, corresponding to an $SU(3)$ spin chain, with spin on each site belongs to either $\mathbf{3}$ or $\bar{\mathbf{3}}$. The Hilbert space at each site is a 6-dimensional (reducible) representation space V of $SU(3)$: $V = \mathbf{3} \oplus \bar{\mathbf{3}}$, and the dimensionality of the Hilbert space of the spin chain is thus reduced to 6^L .

In order to write ADM of operators (7) in compact form, and to facilitate later comparison with the $\mathcal{N} = 4$ mother SYM theory, we introduce the projection operators J^\pm , which projects V to its invariant sub-spaces $\mathbf{3}$ and $\bar{\mathbf{3}}$, respectively; in component form, they projects a vector $v^i \equiv (v^a, \bar{v}^{\bar{a}})$ to its components v^a and to $\bar{v}^{\bar{a}}$. We also define the permutation operator P and the trace operator K , which act on the tensor product $V \otimes V$, respectively, as

$$\begin{aligned} P(u \otimes v) &= v \otimes u, \\ K(u \otimes v) &= (u \cdot v) \sum_a (\hat{e}_a \otimes \hat{e}_{\bar{a}} + \hat{e}_{\bar{a}} \otimes \hat{e}_a), \end{aligned} \quad (16)$$

where u, v are vectors in V and $(u \cdot v) = \sum_a (u^a v^{\bar{a}} + u^{\bar{a}} v^a)$; moreover, \hat{e}_a and $\hat{e}_{\bar{a}}$ are vectors of an orthonormal basis in V .

Then we write the ADM defined in Eq. (15) as follows:

$$\begin{aligned} \Gamma_{A_1 \dots A_L} &= \sum_{l=1}^L \sum_{i,j=\pm} \Gamma_{l,l+1}^{ij}(A_l) J_l^i J_{l+1}^j, \\ \Gamma_{l,l+1}^{++}(A_l) &= \Gamma_{l,l+1}^{--}(A_l) = -\gamma_{A_l} + \frac{\lambda_h}{8\pi^2} (1 - P_{l,l+1}), \\ \Gamma_{l,l+1}^{+-}(A_l) &= -\gamma_{A_l} + \frac{\lambda_{A_l^+}}{16\pi^2} \sum_{l=1}^L (2 + K_{l,l+1} - K_{l,l+1} P_{l,l+1}) - \frac{\lambda_h}{8\pi^2} \sum_{l=1}^L (1 - K_{l,l+1}) P_{l,l+1}, \\ \Gamma_{l,l+1}^{-+}(A_l) &= -\gamma_{A_l} + \frac{\lambda_{A_l^-}}{16\pi^2} \sum_{l=1}^L (2 + K_{l,l+1} - K_{l,l+1} P_{l,l+1}) - \frac{\lambda_h}{8\pi^2} \sum_{l=1}^L (1 - K_{l,l+1}) P_{l,l+1}. \end{aligned} \quad (17)$$

We note that though the coefficients on the right side depend on the gauge indices A_l , but the operators $P_{l,l+1}$ and $K_{l,l+1}$ act only on the $SU(3)$ indices.

Since at planar one-loop level holomorphic (or anti-holomorphic) operators, consisting of $\mathbf{3}$'s (or of $\bar{\mathbf{3}}$'s, respectively), mix only among themselves. So it makes sense to restrict

the ADM to the holomorphic (or anti-holomorphic) sector, and then to ask whether the so-restricted ADM can be identified with the Hamiltonian of an integrable spin chain or not. (From the quiver diagram, one can easily see that gauge invariance requires (anti-)holomorphic operators have length $L = 3k$ with integer k .)

From above equations we see that the ADM Γ_h , restricted in the holomorphic sector and associated with the projection operators $J_1^- \cdots J_L^-$, is given by

$$\Gamma_{(h)A_1 \cdots A_L} = - \sum_{l=1}^L \gamma_{A_l} + \frac{\lambda_h}{8\pi^2} \sum_{l=1}^L (1 - P_{l,l+1}). \quad (18)$$

A same expression can be obtained for anti-holomorphic operators.

It is worthy to note that in Eq. (18), the dependence on gauge indices A_l can actually be eliminated, since gauge invariance requires that $L = 3k$ and the gauge index sequence goes around the triangular quiver diagram, Fig. 1, only in one direction. Therefore we can rewrite the ADM in the holomorphic sector as

$$\begin{aligned} \Gamma_{(h)} &= \Gamma_0 + \Gamma_1, \\ \Gamma_0 &= -\frac{L}{24\pi^2} \left(\sum_{A=1}^3 \lambda_A - 3\lambda_h \right), \\ \Gamma_1 &= \frac{\lambda_h}{8\pi^2} \sum_{l=1}^L (1 - P_{l,l+1}). \end{aligned} \quad (19)$$

It is easy to see that Γ_0 is a constant, while Γ_1 depends only on λ_h , the superpotential coupling. By introducing the spin operators

$$S_{ij}^{ab} = \frac{1}{\sqrt{2}} (\delta_i^a \delta_j^b - \delta_j^a \delta_i^b), \quad (20)$$

for each lattice site the ADM, Γ_1 , can be rewritten in terms of manifest spin-spin interactions:

$$\Gamma_1 = \frac{\lambda_h}{8\pi^2} \sum_{l=1}^L \{ S_l^{ab} S_{l+1}^{ab} - (S_l^{ab} S_{l+1}^{ab})^2 \}. \quad (21)$$

Since only the permutation operator P appears in the ADM (19), it can be identified to be the Hamiltonian of an *integrable* spin chain with $SU(3)$ symmetry, for *arbitrary* 't Hooft couplings λ_A . The detail on the integrability of $SU(M)$ spin chain will be presented in appendix A. The Hilbert space of this spin chain is the tensor product $U_1 \otimes \cdots \otimes U_L$ with $U_l = \mathbb{C}^3$ spanned by covariant vectors of $SU(3)$. Thus, though the full ADM (17) in

our model does not in general correspond to an integrable spin chain, the ADM (19) in the holomorphic sector *does*.

It is interesting to note that in our model, an integrable $SU(3)$ spin chain (19) appears even if the theory is away from the conformal fixed line, independent of the values of λ_A and λ_h . The integrability of the system enable us to find the exact one-loop anomalous dimensions of (anti-)holomorphic operators via applying Bethe ansatz equations. Some examples and computational details are shown in appendix B.

In Sec. II we have shown that at IR the gauge interaction and superpotential interaction are decoupled. In the limit $\lambda_h \rightarrow 0$, the ADM (19) is proportional to identity operator so that (anti-)chiral operators (7) do not mix with each other under planar one-loop renormalization. However, the case with $\lambda_A \rightarrow 0$ is more interesting. When $\lambda_A \rightarrow 0$, the present $\mathcal{N} = 1$ gauge theory actually approaches to an $\mathcal{N} = 1$ Wess-Zumino model. Then, according to the above Eq. (19), the planar one-loop ADM for composite operators consisting of purely (anti-) chiral scalars in this Wess-Zumino model also corresponds to an integrable $SU(3)$ spin chain. (Since the chiral supermultiplets are taken to be bi-fundamental representations of three global $U(N)$ groups, here the planar limit makes sense as the limit in which $N \rightarrow \infty$ with hN^2 kept fixed.) We note that in this Wess-Zumino model there is no gauge invariance constraint, so the length L of the holomorphic composite operators for the integrable ADM (or quantum spin chain) do *not* need to be restricted to a multiple of three!

To conclude, we make the following remark: Though in the above we have argued that the $SU(3)$ spin chain arising from the ADM in the holomorphic sector is "blind" to the discrete index A , the operator mixing is *not*. Because each ϕ -field factor in the composite operators now carries the index A due to orbifolding, as we will see in appendix B, the operator mixing becomes completely different from that for the $SO(6)$ spin chain in $\mathcal{N} = 4$ SYM.

V. GENERALIZATION

A. Lift to the $\mathcal{N} = 4$ SYM

We have mentioned that on the conformal fixed line, our present $\mathcal{N} = 1$ model is a \mathbb{Z}_3 orbifolding of $\mathcal{N} = 4$ SYM [18, 19]. Then all correlation functions of the orbifolded theory

are known to coincide with those of $\mathcal{N} = 4$ SYM, modulo a rescaling of the gauge coupling constant. This has been shown either by using string theory [24], or by using Feynman diagrams in field theory [25]. Thus we expect that the spin Hamiltonian (19) is related to the integrable $SO(6)$ spin Hamiltonian obtained by Minahan and Zarembo [14].

To see this without explicit calculation, we may argue as follows. First notice that for fixed index A , the bi-fundamental ϕ^a fields, together with their conjugate $\phi_{\bar{a}}$, form a 6-dimension vector. They are originated from the \mathbb{Z}_3 orbifold projection of the 6-dimensional anti-symmetric representation of $SU(4)$ R-symmetry in $\mathcal{N} = 4$ SYM:

$$\phi^a = \gamma_g^\dagger ((R_g^6)^a_b \phi^b) \gamma_g, \quad \text{for any } g \in \mathbb{Z}_3. \quad (22)$$

Here γ_g are the regular representation of $g \in \mathbb{Z}_3$ in $U(3N)$ group, R_g^6 the 6-dimensional representation of $g \in \mathbb{Z}_3$ in $SU(4)$ R-symmetry group. In particular, ϕ^a corresponds charge-2/3 fields in **6** of $SU(4)$ under R_g transformation, while $\phi_{\bar{a}}$ charge-(-2/3) ones. According to a statement in ref. [25], the γ_g -action on correlation functions (9) is trivial in the planar limit. Moreover, gauge invariance of a composite operator (7) requires $|L - 2k|/3$ to be integer, where k is the number of $\phi_{\bar{a}}$. Consequently such composite operators have zero charge under R_g , and the ADM or spin-chain Hamiltonian (17) at the conformal points should be the same as the integrable $SO(6)$ spin Hamiltonian in $\mathcal{N} = 4$ SCYM.

Explicitly on the conformal fixed line, one has $\lambda_A = \lambda_h = \lambda$, then the ADM (17) is simplified to

$$\begin{aligned} \Gamma &= \frac{\lambda}{8\pi^2} \sum_{l=1}^L (1 - P_{l,l+1}) (J_l^+ J_{l+1}^+ + J_l^- J_{l+1}^-) \\ &+ \frac{\lambda}{16\pi^2} \sum_{l=1}^L (2 + K_{l,l+1} + K_{l,l+1} P_{l,l+1} - 2P_{l,l+1}) (J_l^+ J_{l+1}^- + J_l^- J_{l+1}^+) \\ &= \frac{\lambda}{16\pi^2} \sum_{l=1}^L (2 + \bar{K}_{l,l+1} - 2P_{l,l+1}). \end{aligned} \quad (23)$$

Here in the last line, $\bar{K}(a \otimes b) = (a^\dagger \cdot b) \sum_i \hat{e}^i \otimes \hat{e}^i$. The rank-2 anti-symmetric representation of $SU(4)$ is related to the vector representation of $SO(6)$ via an unitary transformation U . Notice that $(U \otimes U) \bar{K} (U \otimes U)^\dagger = K$, the ADM (23) is indeed the Hamiltonian of the same integrable $SO(6)$ spin chain as in ref. [14].

B. Other $\mathcal{N} = 1$ SYM and Wess-Zumino model

We have shown that, in our present $\mathcal{N} = 1$ model the ADM of (anti-)holomorphic composite operators corresponds to a Hamiltonian of integrable $SU(3)$ spin chain. One may wonder whether this result can be generalized to other supersymmetric models with global $SU(M)$ symmetry, since the integrability condition for a holomorphic spin chain with any $SU(M)$ symmetry is quite simple, involving only the identity and permutation operators.

For a general $\mathcal{N} = 1$ gauge theory with M chiral superfields, our basic observation is the following: The contributions of diagrams Fig 2a and 2c are always proportional to the identity matrix in $SU(M)$ indices. The quartic scalar vertex in Fig 2b from the gauge interaction (the first term of the Lagrangian (2)) can be effectively treated as an exchange of the D -component of the vector superfield, so its contribution is also proportional to the identity matrix in $SU(M)$ indices. The non-trivial $SU(M)$ structure in planar one-loop ADM can only come from the quartic scalar vertex in Fig. 2b with superpotential interactions (the third term in eq. (2)).

Let us first consider a $\mathcal{N} = 1$ $SU(N)$ gauge theory, which has M chiral superfields transforming as \mathbf{M} of a global $SU(M)$ group and as a certain representation \mathcal{R} under gauge group. In order to large N limit makes sense for superpotential coupling yet, we restrict the representation \mathcal{R} in which degrees of freedom of chiral superfields are order to N^2 at least. In addition to gauge invariance, the superpotential is required to be $SU(M)$ -invariant and of degree three (for renormalizability). Therefore the superpotential has to contain an $SU(M)$ -invariant tensor of rank 3. Consequently the only possibility is $M = 3$ and the superpotential contains the $SU(3)$ -invariant tensor ϵ_{abc} . It is not hard to see that at the planar one-loop level, the ADM of (anti-)holomorphic composite operators contains only the identity and permutation operator P and, therefore, can be identified with the Hamiltonian of an integrable $SU(3)$ spin chain.

Now let take the limit in which the 't Hooft coupling tends to zero, then this model becomes a Wess-Zumino model with three chiral superfields, each transforming as representation \mathcal{R} of $SU(N)$ group and together as $\mathbf{3}$ of $SU(3)$ flavor group. With an $SU(3)$ -invariant superpotential, the ADM of the (anti-)holomorphic composite operators again correspond to an integrable $SU(3)$ spin chain, in the same way as the limit we discussed at the end of Sec. IV.

C. An $\mathcal{N} = 2$ SYM

If the SYM has an extra $U(1)$ R-symmetry, the story will become a little bit complicated. As an example, let us consider the orbifolded $\mathcal{N} = 2$ model proposed in [26]. The gauge group of the model is $SU(N)^{(1)} \times SU(N)^{(2)} \cdots SU(N)^{(K)}$. The bosonic fields in the vector multiplet are denoted as $(A_{\mu I}, \phi_I)$ with $I = 1, \dots, K$. The matter fields are hypermultiplets B_I^a , ($a = 1, 2$), where B_I belongs to the bi-fundamental representation $(N^{(I-1)}, \bar{N}^{(I)})$ of the $(I-1)$ -th and I -th gauge groups, and they form a doublet (labeled by the index a) under R-symmetry $U(1)_R \times SU(2)_R$. In order to construct a renormalizable, gauge-invariant and $SU(2)_R$ -invariant superpotential, one has to require $K = 2$. The superpotential is of the form

$$\mathcal{W} = \frac{h}{2} \epsilon_{ab} \text{tr}(B_1^a B_2^b \Phi_1 + B_2^a B_1^b \Phi_2), \quad (24)$$

where $a, b = 1, 2$ are $SU(2)_R$ indices.

We can form three types of gauge-invariant, holomorphic composite operators \mathcal{O}_i , ($i = 1, 2, 3$) with L sites, which are closed under planar one-loop renormalization: \mathcal{O}_1 consisting of the doublet B_I s only, \mathcal{O}_2 of ϕ_I only and \mathcal{O}_3 with mixed B_I and ϕ_I . The closure and holomorphy of \mathcal{O}_2 restricts one-loop planar diagram corrections to \mathcal{O}_2 to be always diagonal. Its anomalous dimension is

$$\Gamma_{2,I} = \frac{L}{8\pi^2} (\lambda_h - \lambda_I), \quad (25)$$

where λ_I denotes the 't Hooft coupling of the I th gauge group. It is interesting to note that at conformal points, $\lambda_I = \lambda_h$, the operator \mathcal{O}_2 is protected.

The ϵ_{ab} in the superpotential (24) forbids the trace operator K to appear in the ADM of operators \mathcal{O}_1 . Moreover, the quartic scalar vertex derived from the superpotential (24) contains a term $(B_1^1 B_2^2 - B_1^2 B_2^1)(\bar{B}_2^2 \bar{B}_1^1 - \bar{B}_2^1 \bar{B}_1^2)$. The permutation operator P appears, due to this term, in the ADM of \mathcal{O}_1 :

$$\Gamma_{1,I_1} = \frac{1}{8\pi^2} \sum_{l=1}^L (2\lambda_h - \lambda_{I_l} - \lambda_h P_{l,l+1}). \quad (26)$$

Here it should be noticed that I_l ($l > 1$) is uniquely determined by I_1 due to gauge invariance. The ADM in eq. (26) can again be regarded as the Hamiltonian of an integrable spin chain with $SU(2)$ symmetry.

The case for \mathcal{O}_3 is more complicated, since a ϕ_I field and a neighboring B_J may exchange in planar one-loop diagrams. Effectively we can treat an \mathcal{O}_3 operator as insertions of ϕ 's in a \mathcal{O}_1 operator. Noticing an \mathcal{O}_3 consists of L B_I s and k ϕ s is closed with fixed L and k , we can write down its ADM explicitly. For example, let us consider a ϕ_{I_i} inserting at i -th site of \mathcal{O}_1 , so that we get operator \mathcal{O}_3 with $L + 1$ sites which consists of L B_I s and a ϕ . Its ADM can be written as

$$\Gamma_{3,(I_1 \dots I_i \dots I_L)} = \frac{1}{8\pi^2} \sum_{l=1}^{L+1} (2\lambda_h - \lambda_{I_l}) - \frac{\lambda_h}{8\pi^2} \sum_{\substack{l=1, \\ l \neq i-1, i}}^{L+1} P_{l,l+1} - \frac{\lambda_h}{8\pi^2} (P'_{i-1,i} + P'_{i,i+1}), \quad (27)$$

where the permutation operator P' exchange ϕ_{I_i} and neighboring B_I .

We can see the ADM (27) is similar to the ADM (26), if we suppress the gauge group indices. Therefore, we can formally define a Hilbert space $\mathcal{H} = \prod_{l=1}^L \otimes V$ with $V = \mathbb{C}^2 \oplus \mathbb{C}$. That is, to put B_I^a and ϕ_I into a triplet and to consider the set of gauge invariant operators $\{\mathcal{O}\} = \{\mathcal{O}_1\} \oplus \{\mathcal{O}_2\} \oplus \{\mathcal{O}_3\}$. The ADM for this set of operators is

$$\Gamma_{I_1} = \frac{1}{8\pi^2} \sum_{l=1}^L (2\lambda_h - \lambda_{I_l} - \lambda_h P_{l,l+1}). \quad (28)$$

The above ADM can be regarded as the Hamiltonian of an integrable spin chain with $SU(3)$ (instead of $SU(2)$) symmetry. It implies that in the ADM of holomorphic composite scalar operators, its symmetry (as spin chain) is enhanced from $SU(2)$ R -symmetry to $SU(3)$, if gauge group indices are suppressed, as allowed by gauge invariance constraint. The difference between the Hamiltonian (28) and (19) is only that they have different constant terms.

Finally, since this $\mathcal{N} = 2$ model is obtained by deforming an \mathbb{Z}_2 orbifolding of $\mathcal{N} = 4$ SYM, the ADM for gauge invariant composite scalar operators coincides with that in $\mathcal{N} = 4$ SYM at conformal points. The same as in the $\mathcal{N} = 1$ case, it corresponds to the Hamiltonian of an integrable spin chain with $SO(6)$ symmetry. Consequently for the ADM of SYM theories we obtain a cascade of integrable structures from orbifolding (or taking quotient of) $\mathcal{N} = 4$ SYM (see Fig. 3).

D. A conjecture

In the above examples we have seen that in these orbifolded daughters of $\mathcal{N} = 1, 2$ SYM, the variation of 't Hooft couplings affects only the total sum of constant terms in

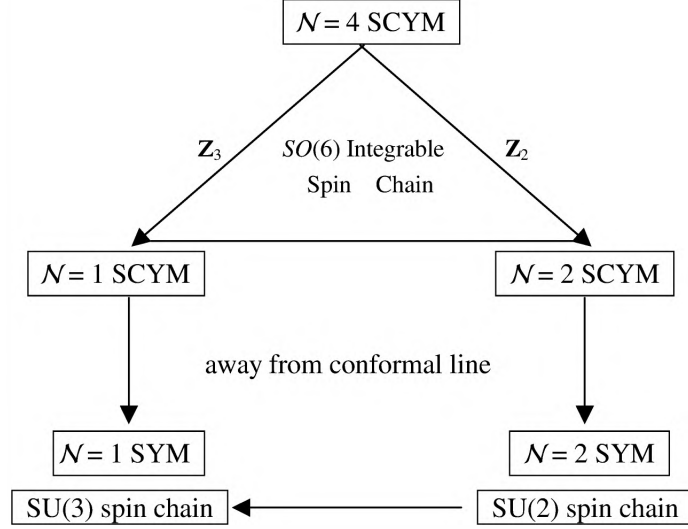


FIG. 3: The cascade of integrable structure in SYM.

the Hamiltonian of the quantum spin chain, and the change by an overall factor in the superpotential couplings only leads to an overall factor for the spin chain coupling. Neither of them affect the integrability of the spin chain Hamiltonian. So it is the symmetry structure of the superpotential that dictates the integrable structure and, moreover, the integrable structure should exist for all deformed orbifolded daughters of $\mathcal{N} = 4$ SYM with unbroken supersymmetries.

More precisely, we are led to make the following conjecture: In *all* $\mathcal{N} = 1, 2$ orbifolded daughters [18, 19] of the $\mathcal{N} = 4$ SYM, (with a non-abelian global symmetry), with their 't Hooft and superpotential couplings deformed away from the conformal points while keeping the global symmetry of the superpotential, there is always a nontrivial *integrable structure* in operator mixing at planar one-loop level for (anti-) holomorphic composite operators, at least for those consisting of purely scalars without derivatives. Moreover, this integrable structure survives in the resulting Wess-Zumino models when all 't Hooft couplings are sent to zero.

VI. SUMMARY

We have presented examples in which an integrable structure appears in four dimensional $\mathcal{N} = 1, 2$ super Yang-Mills theories or Wess-Zumino models, even with conformal symmetry broken. In these examples the planar one-loop anomalous dimension matrix for composite

operators consisting of (anti-)holomorphic scalars can be written as the Hamiltonian of an integrable spin chain with $SU(3)$ (or $SU(2)$) symmetry. It indicates that the origin of the integrable structure in four dimensional Yang-Mills theories is of more profound origin, not restricted to superconformal symmetry. Though these spin chains can be formally viewed as a subsector of the integrable $SO(6)$ chain found in $\mathcal{N} = 4$ SCYM[14], as will be shown in appendix B, the operator mixing in our models is completely different, because each scalar field factor in the composite operators now carries an extra discrete index A , which is absent in $\mathcal{N} = 4$ SYM. Moreover, the appearance of an integrable structure in various SYM theories enables us to use methods in one-dimensional integrable models to study properties of SYM. In appendix A and B below, we will show some examples for how to find one-loop anomalous dimensions of composite operators of SYM via solving Bethe ansatz equations.

Though the study of the ADM of composite operators in $\mathcal{N} = 4$ SYM was first motivated by the BMN limit, which is conjectured to be dual to string theory in the pp-wave background, the integrability of the ADM, as a spin chain Hamiltonian, has nothing to do with the BMN limit. All our examples confirm this point once more.

As we have seen both in the $\mathcal{N} = 4$ SYM and our models with $\mathcal{N} = 1, 2$, the length of the integrable spin chain can be finite. Moreover we have shown that non-trivial integrable structure of these spin chain is actually encoded in the superpotential interactions, rather than gauge interactions. In the cases we studied, it is the symmetry of the superpotential that determines the symmetry of the integrable spin chain, and it is the superpotential coupling appears in the non-trivial part of the integrable Hamiltonian, even though our deformation of the superpotential strength has made the theory away from the conformal points.

Recently there are further developments on the $\mathcal{N} = 4$ SYM integrable super spin chain: A study on the two-loop corrections to the integrable Hamiltonian yielded intriguing evidence that the higher order corrections do not break the integrability property[28]. In addition, it was shown that there is a relation between the infinite-dimensional non-local symmetry of type IIB superstring in $AdS_5 \times S^5$ and a non-abelian and nonlinear infinite-dimensional Yangian algebra for weakly coupled SCYM [29, 30]. It would be interesting to address similar issues in the framework of $\mathcal{N} = 1, 2$ supersymmetric models, such as those we have presented in this paper. Moreover, from the string theory point of view, the following questions arise naturally: Is some of the gauge theories studied here dual to gravity or string theory? Could

the integrable structure in the supersymmetric gauge theories be originated from or related to string theory?

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APPENDIX A: GENERALIZED HEISENBERG (ANTI-)FERROMAGNET

We will show that the spin Hamiltonian in Eq. (19) is just one of the generalized Heisenberg ferromagnet models discussed in ref.[31]. For readers' convenience, in this appendix we give a brief review on the generalized Heisenberg (anti-)ferromagnet — a quantum system with M components on a one-dimensional chain with nearest-neighbor (short-range) interactions. The complete space of state is

$$\mathcal{H} = \bigotimes_{l=1}^L \mathcal{H}_l, \quad \mathcal{H}_l = \mathbb{C}^M. \quad (\text{A1})$$

The Lax operator, associated with the n -th site on the chain, $L_{n,a}(\mu)$, acts on the tensor space $\mathcal{H}_n \otimes V_a$ with auxiliary space $V_a = \mathbb{C}^M$. It is of the form

$$L_{n,a}(\mu) = a(\mu)I_{n,a} + b(\mu)P_{n,a}, \quad (\text{A2})$$

where $P_{n,a}$ is the permutation operator acting on $\mathcal{H}_n \otimes V_a$, μ the spectral parameter, and $a(\mu) + b(\mu) = 1$, $a(\mu) = \mu/(\mu + i\varepsilon)$ with $\varepsilon = \pm 1$ corresponding to anti-ferromagnet and ferromagnet respectively.

The transfer matrix $T_{L,a}(\mu)$ defined by

$$T_{L,a}(\mu) = L_{L,a}(\mu) \cdots L_{1,a}(\mu) \quad (\text{A3})$$

is a monodromy around a circle (assuming the periodical boundary conditions with $\mathcal{H}_{L+1} = \mathcal{H}_1$ for the chain). It satisfies the following relations

$$R_{ab}(\mu - \nu)T_{L,a}(\mu)T_{L,b}(\nu) = T_{L,b}(\nu)T_{L,a}(\mu)R_{ab}(\mu - \nu), \quad (\text{A4})$$

where $R_{ab}(\mu) = b(\mu) + a(\mu)P_{a,b}$. Taking the trace on the auxiliary spaces V_a and V_b , we get the commutative relation

$$[t(\mu), t(\nu)] = 0, \quad t(\mu) = \text{tr}_a T_{L,a}(\mu). \quad (\text{A5})$$

This allows us to treat $t(\mu)$ as a generating function of commuting conserved quantities:

$$M^{(l)} = i \left(\frac{d}{d\mu} \right)^l \ln [t(\mu)t(0)^{-1}]|_{\mu=0}. \quad (\text{A6})$$

By definition, the momentum operator P on a lattice, $P = -i \ln t(0)$, is related to the discrete shift operator (by one site)

$$t(0)\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_L = \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_L \otimes \mathcal{H}_1. \quad (\text{A7})$$

While the Hamiltonian of the system is

$$H = M^{(1)} = \varepsilon \sum_{l=1}^L (P_{l,l+1} - 1). \quad (\text{A8})$$

Then we can see that for very large L the spin Hamiltonian (19) for our $\mathcal{N} = 1$ model is nothing but Eq. (A8) with $\varepsilon = -1$ (for a ferromagnetic system).

For $\mathcal{H}_n = \mathbb{C}^3$, the eigenvalues $\Lambda(\mu)$ of the operator $t(\mu)$ is [31]

$$\Lambda(\mu) = a(\mu)^L \left\{ \prod_{j=1}^m \frac{1}{a(\mu - \mu_j^{(1)})} + \prod_{i=1}^n \frac{1}{a(\mu - \mu_i)} \prod_{j=1}^m \frac{1}{a(\mu - \mu_j^{(1)})} \right\} + \prod_{i=1}^n \frac{1}{a(\mu_i - \mu)}. \quad (\text{A9})$$

Here the rapidity variables μ_i and $\mu_j^{(1)}$ satisfy the algebraic Bethe Ansatz equations (ABAE):

$$\begin{aligned} \prod_{k=1}^n a(\mu_j^{(1)} - \mu_k) &= \prod_{\substack{k=1 \\ k \neq j}}^m \frac{a(\mu_j^{(1)} - \mu_k^{(1)})}{a(\mu_k^{(1)} - \mu_j^{(1)})}, \\ a(\mu_j)^L \prod_{k=1}^m \frac{1}{a(\mu_k^{(1)} - \mu_j)} &= \prod_{\substack{k=1 \\ k \neq j}}^n \frac{a(\mu_j - \mu_k)}{a(\mu_k - \mu_j)}. \end{aligned} \quad (\text{A10})$$

The eigenvalues of the momentum operator P and Hamiltonian H are, respectively,

$$\begin{aligned} p(\{\mu_j\}) &= \sum_{j=1}^n p(\mu_j) = \frac{1}{i} \sum_{j=1}^n \ln \frac{\mu_j + i\varepsilon}{\mu_j}, \\ E(\{\mu_j\}) &= \sum_{j=1}^n \epsilon(\mu_j) = - \sum_{j=1}^n \frac{\varepsilon}{\mu_j(\mu_j + i\varepsilon)}. \end{aligned} \quad (\text{A11})$$

Introducing new rapidity variables $\mu_{1,j}$ and $\mu_{2,j}$ via

$$\begin{aligned} \mu_j &= \frac{1}{2}\mu_{1,j} - \frac{i}{2}\varepsilon, \\ \mu_j^{(1)} &= \frac{1}{2}\mu_{2,j} - i\varepsilon, \end{aligned} \quad (\text{A12})$$

we get the usual expressions for the ABAE (A10)

$$\begin{aligned} \left(\frac{\mu_{1,j} - i\varepsilon}{\mu_{1,j} + i\varepsilon} \right)^L &= \prod_{\substack{k=1 \\ k \neq j}}^{n_1} \frac{\mu_{1,j} - \mu_{1,k} - 2i\varepsilon}{\mu_{1,j} - \mu_{1,k} + 2i\varepsilon} \prod_{l=1}^{n_2} \frac{\mu_{1,j} - \mu_{2,l} + i\varepsilon}{\mu_{1,j} - \mu_{2,l} - i\varepsilon}, \\ 1 &= \prod_{\substack{k=1 \\ k \neq j}}^{n_2} \frac{\mu_{2,j} - \mu_{2,k} - 2i\varepsilon}{\mu_{2,j} - \mu_{2,k} + 2i\varepsilon} \prod_{l=1}^{n_1} \frac{\mu_{2,j} - \mu_{1,l} + i\varepsilon}{\mu_{2,j} - \mu_{1,l} - i\varepsilon}, \end{aligned} \quad (\text{A13})$$

and for eigenvalues of total momentum and energy, we have

$$\begin{aligned} p &= \sum_{j=1}^{n_1} p(\mu_{1,j}), & p(\mu) &= \frac{1}{i} \ln \frac{\mu - i\varepsilon}{\mu + i\varepsilon}, \\ E &= \sum_{j=1}^{n_1} \epsilon(\mu_{1,j}), & \epsilon(\mu) &= -\frac{4\varepsilon}{\mu^2 + 1}. \end{aligned} \quad (\text{A14})$$

APPENDIX B: ANOMALOUS DIMENSIONS FROM BETHE ANSATZ

In this appendix we present several concrete examples to show how to obtain anomalous dimensions by solving Bethe ansatz equations. The discussion will be similar to that in refs. [14, 32], but there are new issues to address, associated with the discrete index A arising from orbifolding, which is absent in the $\mathcal{N} = 4$ case. In section 4 we have shown that the ADM for holomorphic operators in our $\mathcal{N} = 1$ model gives rise to an $SU(3)$ spin chain, which can be viewed as a closed sector of the $SO(6)$ spin chain in the $\mathcal{N} = 4$ SYM. However, as we will see below, the operator mixing in our model is completely different, because each ϕ -field factor now carries an extra discrete index A that labels the gauge group factors.

The $SU(3)$ symmetry of our spin chain is related to Lie algebra A_2 , which has two simple roots,

$$\vec{\alpha}_1 = \left(\sqrt{\frac{3}{2}}, -\frac{1}{\sqrt{2}}\right), \quad \vec{\alpha}_2 = (0, \sqrt{2}), \quad (\text{B1})$$

and the highest weight vectors that generate the fundamental and anti-fundamental representation are, respectively,

$$\vec{w}_1 = \left(\sqrt{\frac{2}{3}}, 0\right), \quad \vec{w}_2 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right). \quad (\text{B2})$$

We always take $\varepsilon = -1$ in the rest of this appendix. Then the ABAE (A13) follows from the general ABAE [33]:

$$\left(\frac{\mu_{q,j} + i\vec{\alpha}_q \cdot \vec{w}}{\mu_{q,j} - i\vec{\alpha}_q \cdot \vec{w}}\right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{n_q} \frac{\mu_{q,j} - \mu_{q,k} + i\vec{\alpha}_q \cdot \vec{\alpha}_q}{\mu_{q,j} - \mu_{q,k} - i\vec{\alpha}_q \cdot \vec{\alpha}_q} \prod_{q' \neq q} \prod_{l=1}^{n_{q'}} \frac{\mu_{q,j} - \mu_{q',l} + i\vec{\alpha}_q \cdot \vec{\alpha}_{q'}}{\mu_{q,j} - \mu_{q',l} - i\vec{\alpha}_q \cdot \vec{\alpha}_{q'}}. \quad (\text{B3})$$

1. Physics

Each factor in the composite operator corresponds a site in the spin chain, which is occupied by a scalar field, being one component of the $SU(3)$ triplet ϕ^a . For the sake of convenience we denote them by (Z, Y, W) in this appendix.

a] The ground state Ω

Because the eigen-energy of the ferromagnet system (A8), namely Γ_1 in eq. (19), is non-negative, the ground state Ω must have zero eigen-energy and total momentum. Consequently, Ω must consist of only one component of (Z, Y, W) . For convenience we assume it consists of Z only². Since there is no impurities of μ_1 and μ_2 , it is not hard to see that $P_\Omega = E_\Omega = 0$.

In our $\mathcal{N} = 1$ SYM, the anomalous dimension of Ω is given by the "zero-point energy" of the system (Γ_0 in Eq. (19)):

$$\gamma_\Omega = \tilde{E}_0 = -\frac{L}{24\pi^2}(\lambda_1 + \lambda_2 + \lambda_3 - 3\lambda_h). \quad (\text{B4})$$

Along the conformal line, $\lambda_h = \lambda_A$, one has $\gamma_\Omega = 0$, so that the dimension of Ω is indeed protected by superconformal invariance.

[2] In $\mathcal{N} = 4$ SYM this state corresponds to the ground state of the BMN operator.

b] States with impurities

In the present case there are two types of impurities in the spin chain, labeled by two rapidities: $\mu_{1,j}$ and $\mu_{2,j}$, which are associated with the two simple roots $\vec{\alpha}_1$ and $\vec{\alpha}_2$ respectively. The states with impurities correspond to excitations of W and Y (i.e. the replacements of some Z by W and/or Y) in the ground state Ω . Since now both $\vec{w}_1 - \vec{\alpha}_1$ and $\vec{w}_1 - \vec{\alpha}_2$ are also weights, in addition to purely μ_1 or μ_2 impurities, $\mu_1 - \mu_2$ bounded impurities are also allowed. However, the weight $\vec{w}_1 - \vec{\alpha}_2$ is not equivalent to \vec{w}_1 , so it does not lie in the fundamental representation. If we restrict ourselves to holomorphic operators, a single μ_2 -impurity without being bounded to a μ_1 -impurity on the same site is not allowed. The physical interpretation is the following: A single μ_1 -impurity ($\vec{w}_1 - \vec{\alpha}_1$) creates a W replacement in the state Ω , while a $\mu_1 - \mu_2$ bounded impurity ($\vec{w}_1 - \vec{\alpha}_1 - \vec{\alpha}_2$) creates a Y replacement. But an individual μ_2 -impurity ($\vec{w}_1 - \vec{\alpha}_2$) would create a \bar{Y} replacement in Ω , breaking the holomorphic nature of the composite operator.

c] The trace condition

An important observation is that the composite operators that we are considering are gauge invariant after taking the trace in the product gauge group $U(N) \times U(N) \times U(N)$, while the scalar fields belong to bi-fundamental representations of two gauge groups, instead of the adjoint of one group. So gauge invariance and holomorphy of the composite operators requires that we are dealing with a chain with length $L = 3k$ with k integer, and

$$t(0)^3 \Psi(\{\mu\}) = \Psi(\{\mu\}). \quad (\text{B5})$$

In other words, after shifting the chain by one site three times in the same direction, we should obtain the same composite operator. Therefore we have the cubic trace condition:

$$\left(\prod_{j=1}^{n_1} \frac{\mu_{1,j} + i}{\mu_{1,j} - i} \right)^3 = 1. \quad (\text{B6})$$

It is easy to verify that this trace condition is consistent with the ABAE (A13). It follows that the total momentum can be only $2n\pi/3$ with n integer.

We note that the condition (B5) or (B6) corresponds to, but is very different from, the trace condition of the $SO(6)$ spin chain in $\mathcal{N} = 4$ SYM [14]. In the latter case, one has power 1 instead of power 3 in Eq. (B5) and Eq. (B6). The reason for power 3 in our model is obviously related to the orbifolding by Z_3 , which leads to an extra three-valued index A for the holomorphic scalars. Below we will see that it is the appearance of this index, though the spin Hamiltonian is "blind" to it, that makes the operator mixing very different from that in $\mathcal{N} = 4$ SYM.

2. One impurity

The first non-trivial and interesting case is a single μ_1 -impurity in Ω . This is the simplest example which shows how and why the operator mixing in our model is very different from that in the $\mathcal{N} = 4$ SYM, though the spin Hamiltonian may be viewed as a closed subsector in the $SO(6)$ spin chain in the latter. In $\mathcal{N} = 4$ SYM, a single impurity with non-zero momentum is not allowed, because of the trace condition. As shown below, however, the present case does allow a single μ_1 -impurity.

The trace condition and the ABAE now reduce to

$$\left(\frac{\mu_1 + i}{\mu_1 - i}\right)^3 = 1. \quad (\text{B7})$$

Here we have used the fact $L = 3k$ with k integer. The above equation yields

$$\begin{aligned} p &= p(\mu_1) = \frac{2n\pi}{3}, \\ E &= \epsilon(\mu_1) = 4 \sin^2 \frac{n\pi}{3}. \end{aligned} \quad (\text{B8})$$

Then anomalous dimension is

$$\gamma_n = \frac{\lambda_h}{2\pi^2} \sin^2 \frac{n\pi}{3} + \gamma_\Omega. \quad (\text{B9})$$

In the language of $\mathcal{N} = 1$ SYM, we are now considering excitations with one W replacement in the ground state Ω . There are *three* distinct possibilities:

$$\begin{aligned} \mathcal{O}_1 &= \text{tr}\{W_1 Z_2 Z_3 (Z_1 Z_2 Z_3)^{k-1}\}, \\ \mathcal{O}_2 &= \text{tr}\{Z_1 W_2 Z_3 (Z_1 Z_2 Z_3)^{k-1}\}, \\ \mathcal{O}_3 &= \text{tr}\{Z_1 Z_2 W_3 (Z_1 Z_2 Z_3)^{k-1}\}, \end{aligned} \quad (\text{B10})$$

where the subscripts 1, 2, 3 are values of the index A , labeling three types of bi-fundamentals. In the above operator basis, the ADM (19) reads

$$\Gamma = \frac{\lambda_h}{8\pi^2} M + \Gamma_0, \quad M = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \quad (\text{B11})$$

The matrix M has eigenvalues $\{3, 3, 0\}$. Therefore we obtain the eigenvalues of Γ as follows:

$$\gamma_1 = \gamma_2 = \frac{3\lambda_h}{8\pi^2} + \gamma_\Omega, \quad \gamma_3 = \gamma_\Omega. \quad (\text{B12})$$

The results are precisely the same as one obtains from the ABAE. The corresponding eigenvectors are $\{a\mathcal{O}_1, b\mathcal{O}_2, -(a+b)\mathcal{O}_3\}$ for $\gamma_1 = \gamma_2$ and $c\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$ for γ_3 with arbitrary constants a, b, c .

3. Two impurities

Two impurities can be either two μ_1 or one μ_1 and one μ_2 . Let us consider two μ_1 -impurities first, i.e. excitations with two W replacements in the ground state. The trace condition and the ABAE now reduce to

$$\begin{aligned} \frac{\mu_{1,1} + i}{\mu_{1,1} - i} \cdot \frac{\mu_{1,2} + i}{\mu_{1,2} - i} &= e^{2in\pi/3}, \\ \left(\frac{\mu_{1,1} + i}{\mu_{1,1} - i} \right)^L &= \frac{\mu_{1,1} - \mu_{1,2} + i}{\mu_{1,1} - \mu_{1,2} - i}. \end{aligned} \quad (\text{B13})$$

For $n = 0$, we must impose $\mu_{1,1} = -\mu_{1,2}$, then only real solutions are allowed. We get the momenta $p(\mu_{1,1}) = 2m\pi/(L-1)$ and the anomalous dimensions (with m an integer):

$$\gamma_m^{(n=0)} = \frac{\lambda_h}{\pi^2} \sin^2 \frac{m\pi}{L-1} + \gamma_\Omega. \quad (\text{B14})$$

For $n = 1, 2$, however, both of real and complex solutions are allowed. If $\mu_{1,1}$ is real, from Eq. (B13) we obtain

$$(L-1)\vartheta(\mu_{1,1}) = m\pi + \vartheta(\mu_{1,1} \mp \frac{2}{\sqrt{3}}), \quad \mu_{1,2} = \frac{\sqrt{3} \pm \mu_{1,1}}{\sqrt{3}\mu_{1,1} \mp 1}, \quad (\text{B15})$$

where $\vartheta(x) \equiv \arctan(x)$, the integer m parameterizes different branches of the logarithmic function. Eq. (B15) in general can not be solved analytically. For very large L , however,

we have approximately $\theta = \vartheta(\mu_{1,1}) \simeq m\pi/L$. Consequently the anomalous dimensions are given by

$$\begin{aligned}\gamma_m^{(n=1,2)} &= \frac{\lambda_h}{2\pi^2} \left(1 + \frac{1}{4} \cos 2\theta \mp \frac{\sqrt{3}}{4} \sin 2\theta\right) + \gamma_\Omega \\ &= \frac{\lambda_h}{2\pi^2} \left(1 + \frac{1}{4} \cos \frac{2m\pi}{L} \mp \frac{\sqrt{3}}{4} \sin \frac{2m\pi}{L}\right) + \gamma_\Omega + O\left(\frac{1}{L^2}\right).\end{aligned}\quad (\text{B16})$$

Now let us consider complex solutions. Notice that for $L \rightarrow \infty$, the LHS of the second equation in Eq. (B13) grows (or decreases) exponentially if $\text{Im}\mu_{1,1} \neq 0$. Hence the RHS of this equation together with the first equation in Eq. (B13) lead to the solutions

$$\begin{cases} \mu_{1,1} = \pm \frac{2}{\sqrt{3}} + i, \\ \mu_{1,2} = \pm \frac{2}{\sqrt{3}} - i, \end{cases}, \quad \text{or} \quad \begin{cases} \mu_{1,1} = \pm \frac{2}{\sqrt{3}} - i, \\ \mu_{1,2} = \pm \frac{2}{\sqrt{3}} + i, \end{cases}, \quad (\text{B17})$$

and the anomalous dimension

$$\tilde{\gamma} = \frac{3\lambda_h}{16\pi^2} + \gamma_\Omega. \quad (\text{B18})$$

Notice that $\tilde{\gamma} < \min(\gamma_m^{(n=1,2)})$. It indicates that complex solutions correspond to bound states.

Next we consider the bounded impurity of one μ_1 and one μ_2 . The ABAE together with trace condition now reduce to Eq. (B7) plus

$$\frac{\mu_2 - \mu_1 - i}{\mu_2 - \mu_1 + i} = 1. \quad (\text{B19})$$

The solution from those equations are $\mu_1 = \cot \frac{n\pi}{3}$ and $\mu_2 = \infty$. It yields the following anomalous dimension

$$\gamma_n = \frac{\lambda_h}{8\pi^2} \epsilon(\mu_1) + \gamma_\Omega = \frac{\lambda_h}{2\pi^2} \sin^2 \frac{n\pi}{3} + \gamma_\Omega. \quad (\text{B20})$$

The result is the same as in Eq. (B9). It just reflects the fact that replacing a Z in the ground state either by W or by Y leads to the same anomalous dimension.

4. The highest excited state

For a finite chain the excited state with the highest energy contains as many as possible impurities. Taking the logarithm of the ABAE (A13) we have

$$\vartheta(\mu_{1,j}) = \frac{j\pi}{L} + \frac{1}{L} \sum_{k \neq j}^{n_1} \vartheta\left(\frac{\mu_{1,j} - \mu_{1,k}}{2}\right) - \frac{1}{L} \sum_{k=1}^{n_2} \vartheta(\mu_{1,j} - \mu_{2,k}),$$

$$0 = \frac{j\pi}{L} + \frac{1}{L} \sum_{k \neq j}^{n_2} \vartheta\left(\frac{\mu_{2,j} - \mu_{2,k}}{2}\right) - \frac{1}{L} \sum_{k=1}^{n_1} \vartheta(\mu_{2,j} - \mu_{1,k}). \quad (\text{B21})$$

where we have used that fact that the discreteness of the Bethe roots requires them to be pushed to different branches of the logarithm function. In general the ABAE (B21) can not be solved analytically with more than two impurities. In the thermodynamical limit $L \rightarrow \infty$, however, the Bethe ansatz equations are simplified significantly [34]. In this limit j/L is replaced by a continuous variable x , and the ABAE (B21) is replaced by a set of integral equations:

$$\begin{aligned} \vartheta(\mu_1(x)) &= \pi x + \int dy \vartheta\left(\frac{\mu_1(x) - \mu_1(y)}{2}\right) - \int dy \vartheta(\mu_1(x) - \mu_2(y)), \\ 0 &= \pi x + \int dy \vartheta\left(\frac{\mu_2(x) - \mu_2(y)}{2}\right) - \int dy \vartheta(\mu_2(x) - \mu_1(y)). \end{aligned} \quad (\text{B22})$$

Taking derivatives with respect to μ_1 and μ_2 , we have

$$\begin{aligned} \frac{1}{\mu_1^2 + 1} &= \pi \rho_1(\mu_1) + \int_{-\infty}^{\infty} d\mu' \frac{2\rho_1(\mu')}{(\mu_1 - \mu')^2 + 4} - \int_{-\infty}^{\infty} d\mu' \frac{\rho_2(\mu')}{(\mu_1 - \mu')^2 + 1}, \\ 0 &= \pi \rho_2(\mu_2) + \int_{-\infty}^{\infty} d\mu' \frac{2\rho_2(\mu')}{(\mu_2 - \mu')^2 + 4} - \int_{-\infty}^{\infty} d\mu' \frac{\rho_1(\mu')}{(\mu_2 - \mu')^2 + 1}, \end{aligned} \quad (\text{B23})$$

where the densities ρ_1 and ρ_2 are defined by

$$\rho_1(\mu_1) = \frac{dx}{d\mu_1(x)}, \quad \rho_2(\mu_2) = \frac{dx}{d\mu_2(x)}. \quad (\text{B24})$$

The equations (B23) can be solved by means of Fourier transformation. The results are as follows:

$$\begin{aligned} \rho_1(x) &= \int \frac{dk}{2\pi} e^{ikx} \frac{2 \cosh k}{4 \cosh^2 k - 1}, \\ \rho_2(x) &= \int \frac{dk}{2\pi} e^{ikx} \frac{1}{4 \cosh^2 k - 1}. \end{aligned} \quad (\text{B25})$$

This yields

$$\int_{-\infty}^{\infty} dx \rho_1(x) = \frac{2}{3}, \quad \int_{-\infty}^{\infty} dx \rho_2(x) = \frac{1}{3}. \quad (\text{B26})$$

Consequently we have $2L/3$ μ_1 -impurities and $L/3$ μ_2 -impurities for the highest excited state. They fill all sites on the ferromagnet chain. In our $\mathcal{N} = 1$ SYM, it implies that there are equal number of W , Y and Z scalar fields in the composite operator. Recalling $L = 3k$

with k integer and denoting the $SU(3)$ triplet as ϕ^a , ($a = 1, 2, 3$) again, the operator has the following form:

$$\mathcal{O} \sim \text{tr}(\epsilon_{abc}\phi^a\phi^b\phi^c)^k. \quad (\text{B27})$$

It is an $SU(3)$ singlet with zero total momentum.

The anomalous dimension of this operator is

$$\gamma = \frac{\lambda_h}{2\pi^2} \int_{-\infty}^{\infty} dx \frac{\rho_1(x)}{x^2 + 1} + \gamma_\Omega = \frac{\lambda_h}{24\pi^2} L \left(\frac{\pi}{\sqrt{3}} + 3 \ln 3 \right) + \gamma_\Omega. \quad (\text{B28})$$

It grows linearly with L , the length of the chain.

Similar to the procedure in [14, 32], we can also calculate the anomalous dimensions of operators with a few W and Y replacements. But let us stop here.

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- [1] L.L. Chau, M.L. Ge and Y.S. Wu, *Kac-Moody Algebra in the Self-dual Yang-Mills Equation*, Phys. Rev. **D25** (1982) 1086;
L.L. Chau and Y.S. Wu, *More about Hidden Symmetry Algebra for the Self-Dual Yang-Mills System* Phys. Rev. **D26** (1982) 3581;
L.L. Chau, M.L. Ge, A. Sinha and Y.S. Wu, *Hidden Symmetry Algebra for Self-Dual Yang-Mills Equations* Phys. Lett. **121B** (1983) 391.
 - [2] L. Dolan, *A New Symmetry Group of Real Selfdual Yang-Mills*, Phys. Lett. **113B** (1982) 273.
 - [3] H.C. Tze and Y.S. Wu, *Infinite Number of Local Conservation Laws for the $SU(2)$ Self-dual Yang-Mills Systems*, Nucl. Phys. **B204** (1982) 118.
 - [4] K. Ueno and Y. Nakamura, *Transformation Theory for (Anti)Self-dual Equations and the Riemann-Hilbert Problem*, Phys. Lett. **109B** (1982) 273.
 - [5] J. Avan, H.J. de Vega and J.M. Maillet. *Conformally Covariant Linear System for the Four-Dimensional Self-Dual Yang-Mills Theory*, Phys. Lett. **171B** (1986) 255.
 - [6] W.A. Bardeen, *Selfdual Yang-Mills Theory, Integrability and Multiparton Ampilitudes*, Prog. Theor. Phys. Suppl. **bf 123** (1996) 1.
 - [7] A.D. Popov and C.R. Preitschopf, *Conformal Symmetries of the Self-Dual Yang-Mills Equations*, Phys. Lett. **B374** (1996) 71; A.D. Popov, *Self-Dual Yang-Mills: Symmetries and Moduli Space*, Rev. Math. Phys. **11** (1999) 1091.

- [8] L.N. Lipatov, *High-energy Asymptotics of Multicolor QCD and Exactly Solvable Lattice Models*, JETP Lett. **59** (1994) 596.
- [9] L.D. Faddeev and G.P. Korchemsky, *High-energy QCD as a Completely Integrable Model*, Phys. Lett. **B342** (1995) 311.
- [10] A.V. Belitsky, A.S. Gorsky and G.P. Korchemsky, *Gauge / String Duality For QCD Conformal Operators*, Nucl. Phys. **B667** (2003) 3.
- [11] A.V. Kotikov, L.N. Lipatov and V.N. Velizhanin, *Anomalous Dimensions of Wilson Operators in $\mathcal{N} = 4$ SYM theory*, Phys. Lett. **B557** (2003) 114.
- [12] V.M. Braun, S.E. Derkachov and A.N. Manashov, *Integrability Of Three Particle Evolution Equations In QCD*, Phys. Rev. Lett. **81** (1998) 2020; V.M. Braun, S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Baryon Distribution Amplitudes In QCD*, Nucl. Phys. **B553** (1999) 355; A.V. Belitsky, *Renormalization Of Twist - Three Operators And Integrable Lattice Models*, Nucl. Phys. **B574** (2000) 407; S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Evolution Equations For Quark Gluon Distributions In Multicolor QCD And Open Spin Chain*, Nucl. Phys. **B566** (2000) 203.
- [13] F.A. Dolan and H. Osborn, *Superconformal Symmetry, Correlation Functions And The Operator Product Expansion*, Nucl. Phys. **B629** (2002) 3.
- [14] J.A. Minahan and K. Zarembo, *The Bethe-Ansatz for $\mathcal{N} = 4$ Super Yang-Mills*, JHEP **03** (2003) 013.
- [15] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in Flat Space and PP-wave from $\mathcal{N} = 4$ Super Yang Mills*, JHEP **04** (2002) 013.
- [16] M. Douglas and G. Moore, *D-branes, Quivers, and ALE Instantons*, hep-th/9603167.
- [17] J. Gomis and H. Ooguri, *Penrose Limit of $\mathcal{N} = 1$ Gauge Theories*, Nucl. Phys. **B635** (2002) 106.
- [18] S. Kachru and E. Silverstein, *4-D Conformal theories and Strings on Orbifolds*, Phys. Rev. Lett. **80** (1998) 4855.
- [19] A. Lawrence, N. Nekrasov and C. Vafa, *On Conformal Theories in Four Dimensions*, Nucl. Phys. **B533** (1998) 199.
- [20] M. Douglas, B. Greene and D. Morrison, *Orbifold Resolution by D-branes*, Nucl. Phys. **B506** (1997) 84.
- [21] J.M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv.

- Theor. Math. Phys. **2** (1998) 231.
- [22] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, *Exact Gell-Mann-Low Function of Supersymmetric Yang-Mills Theories from Instanton Calculations*, Nucl. Phys. **B229** (1986) 381; *Beta Function in Supersymmetric Gauge Theories: Instanton versus Traditional Approach*, Phys. Lett. **B166** (1986) 329.
- [23] R.G. Leigh and M.L. Strassler, *Exactly Marginal Operators and Duality in Four Dimensional $\mathcal{N} = 1$ Supersymmetric Gauge Theory*, Nucl. Phys. **B447** (1995) 95.
- [24] M. Bershadsky, Z. Kakushadze and C. Vafa, *String Expansion as Large N Expansion of Gauge Theories*, Nucl. Phys. **B523** (1998) 59;
Z. Kakushadze, *Gauge Theories from Orientifolds and Large N limit*, Nucl. Phys. **B529**(1998) 157.
- [25] M. Bershadsky, A. Johansen, *Large N Limit of Orbifold Field Theories*, Nucl. Phys. **B536** (1998) 141.
- [26] S. Mukhi, M. Rangamani and E. Verlinde, *String from Quivers, Membranes from Moose*, JHEP **05** (2002) 023.
- [27] N. Beisert and M. Staudacher, *The $\mathcal{N} = 4$ SYM Integrable Super Spin Chain*, Nucl. Phys. **B670** (2003) 439.
- [28] N. Beisert, C. Kristjansen and M. Staudacher, *The Dialtation Operator of Conformal $\mathcal{N} = 4$ Super Yang-Mills Theory*, Nucl. Phys. **B664** (2003) 131.
- [29] L. Dolan, C.R. Nappi and E. Witten, *A Relation between Approachs to Integrabilities in Superconforla Yang-Mills Theory*, hep-th/0308089.
- [30] G. Arutyunov, M. Staudacher, *Matching Higher Conserved Charges for Strings and Spins*, hep-th/0310182.
- [31] P.R. Kulish and N.Yu. Reshetikin, *Generalized Heisenberg Ferromagnet and the Gross-Neveu model*, Sov. Phys. JETP **53** (1981) 108.
- [32] L.D. Faddeev, *How Algebraic Bethe Ansatz Works for Integrable Model*, hep-th/9605187.
- [33] E. Ogievetsky and P. Wiegmann, *Factorized S -matrix and the Bethe Ansatz for Simple Groups*, Phys. Lett. **B168** (1986) 360.
- [34] C.N. Yang and C.P. Yang, *J. Math. Phys.* **10** (1969) 1115.